THE
MATHEMATICAL GAZETTE.

EDITED BY
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WITH THE CO-OPERATION OF
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THE LATE CHARLES SAMUEL JACKSON, ESQ., M.A.*

THE news of the sudden decease of the above named gentleman, who was a mathematical master at the "Shop" for a period of more than a quarter of a century, will be received with feelings of profound regret by many officers of the Royal Artillery and Royal Engineers.

Mr. Jackson left evening study (7.15) on the 17th October in his usual health, but was picked up unconscious just outside the West Lodge at 7.30 p.m., and conveyed to the Plumstead Infirmary, where he died the morning after without having regained consciousness.

Nothing was known of the sad occurrence in the "Shop" until after his death. So passed away one who was universally respected and esteemed, and who was held in great regard by all who knew him, both from his refined and gentle character and high intellectual attainments.

The deceased leaves a widow and nine children in very straitened circumstances, as the Government does not provide widow's pension or compassionate allowance as in the case of the family of an officer; it is therefore proposed to invite subscriptions towards raising a sum of money for the benefit of the widow and children, and the undersigned, who have formed themselves into a Committee for this purpose, feel that such an appeal need only be made known in order to secure a prompt and generous response.

Subscriptions may be sent to Messrs. Cox & Co., 16 Charing Cross, S.W., on account of the "Jackson Memorial Fund"—banker's order form which accompanies this leaflet being used for the purpose.

W. F. CLEEVE, Brig.-Gen., Comdt. Royal Military Academy.
H. ROWAN-ROBINSON, Lt.-Col., 2nd in command, R.M. Academy.
W. FOORD-KELCEY, Esq., B.A., Prof. of Math., R.M. Academy.
H. SIMPSON, Capt., Adjutant, R.M. Academy.
D. SMITH, Major, Hon. Secretary and Treasurer.

I venture strongly to urge the importance of a response to this Appeal, which I am sending to many members of the Mathematical Association. C. C. LYNAM, M.A., Oxford.

* We print verbatim the above Appeal, with the conviction that response will be immediate. [W. J. G.]

A PLEA FOR A MORE GENERAL USE OF VECTOR ANALYSIS IN APPLIED MATHEMATICS.

By C. E. Weatherburn, M.A., D.Sc.

That the advantage of using vector analysis in mathematical physics has met with so little recognition by British applied mathematicians must appear rather strange to those of other countries. There are probably not more than half a dozen among us who habitually use vector methods and notation, at any rate in published work. The fact is still more surprising when we recall the work of Maxwell * and Heaviside, seeing that the former gave a sort of authority to the *curl* and *divergence*, while the latter had so much to do with the systematic development of the vectorial calculus.

Perhaps the chief reason † why vector analysis has not come into more general favour with us is that our leaders in applied mathematics have not felt the need of it. It has often been remarked, and perhaps with some degree of truth, that nothing can be accomplished by vector methods that cannot also be done by Cartesian analysis ; and therefore, it is argued, the change is unnecessary and useless. If we had to deal only with minds of special mathematical ability and analytical insight, this conclusion might be accepted. But with the average student so much of his attention is occupied in dealing with the complex array of symbols of partial differentiation to which he is often led in Cartesian analysis, that he is unable to grasp the inner meaning of the work. It is difficult for him in many cases even to see exactly what is expressed in the formulae obtained, involving as they do the three components of a vector quantity in combinations not easy to visualise. And even if the student succeeds in following the argument it is often almost impossible for him to remember either the train of reasoning or the result arrived at. He does not perhaps see why his equations should be differentiated partially with respect to x, y, z and added, or with respect to z, y and subtracted.

Working, however, with the aid of vector analysis, we no longer have three unsymmetrical equations to carry in our thoughts, but a single equation involving only the vector quantity as a whole ; and we thus form the habit of regarding this as one complete quantity rather than as a group of three. The analytical transformations are reduced to a minimum, and the student is able to devote a much greater part of his attention to the meaning of his equations. The reason, too, for a particular step in the analysis is often quite apparent. Suppose, for instance, that we desire to eliminate from our equation a certain vector of which only the *gradient* is present. The obvious course is to take the *curl* of both members. Or, if it is the *curl* of the undesirable vector that is present we immediately take the *divergence* of both sides. The Cartesian equivalent of the former of these eliminations is much more involved.

It is on the strength of personal experience that I advocate the adoption of vector analysis. As a student first at Sydney and then

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* There is no doubt that Maxwell would have gone much further in the vectorial method if he had had ready to hand a system of analysis such as to-day is at our disposal. Unfortunately at that period there was only the theory of quaternions, which he wisely did not employ except in passing reference.

† See, however, the Note at the end of this paper.

at Cambridge, I had some leaning toward the applied subjects; but it was not until the last few years that I gained any knowledge of vector methods.* The newer analysis completely altered my mathematical outlook, and made mathematical physics to me a new subject. An almost dying interest gave place to a keen enthusiasm, and most of the old difficulties vanished in the new light thrown upon the work. It is to save the average students of the coming generation any unnecessary difficulties, and to help them to form clearer conceptions of physical quantities and mathematical processes, that I am advocating the change which has been delayed too long. It is not a matter of vital importance which particular notation we use—whether the dot and cross or the bracket notation, although the former has many advantages. Nor does it make much difference whether we teach in terms of *dyadics* or of *tensors*.

But it is not only the student of average mathematical intelligence who would gain by the change. The vector methods are in themselves so much quicker; and in these days we cannot afford to waste time either in writing or in unnecessary mechanical thought. Associated with this is the great gain in condensation. It is laborious and often irksome to wade through three pages of Cartesian analysis when one page of vectorial would suffice. And the expenditure of mental effort is balanced by no corresponding gain; but rather the shorter analysis affords a clearer mental picture of the meaning of the equations and the significance of the transformations. The argument of condensation is certainly worth serious consideration. In this connection it may be remarked how few English mathematical books are written vectorially. One thinks of a couple of recent works in Relativity, one or two in Mechanics, and Heaviside's Electromagnetic treatise, which can hardly be called recent, but the number is soon exhausted. And yet there is no dearth of excellent treatises on a great variety of subjects. No doubt the chief reason why a larger number of authors have not adopted the vector methods and notation is the lack of familiarity with these methods among their readers. This recalls the practice of "dodging the calculus" so common in elementary text-books, and of course unavoidable when the necessary knowledge is not possessed by the student. Yet every one admits the great advantages to be derived from the use of the infinitesimal calculus, and the early mastery of the elements of it. The case for the vector calculus is exactly parallel; and the student who acquires familiarity with it delights in the facility with which he can make the various transformations, and in the clearer understanding of the main ideas beneath them. To him the more laborious Cartesian analysis appears exactly like dodging the calculus.

It will hardly be seriously objected that to become familiar with the new analysis involves a certain expenditure of time and the effort of learning more formulae. Precisely the same objection may be raised to learning the infinitesimal calculus itself. And, just as in this case, the amount of time and labour saved in the long run is vastly greater than that spent in acquiring the necessary knowledge. Even in the proofs of many of the theorems of the integral calculus vector methods lead to a considerable shortening of detail, and the final results are expressible in forms that are neater and easier to remember than the old ones. Take, for instance, some of the theorems expressing the transformation of integrals connected with lines, surfaces and volumes.

* In undertaking the study of vector analysis, I followed the example of my learned colleague at Melbourne University—Mr. J. H. Michell, F.R.S.—who adopted it several years previously.

In vector notation we have

$$\int_0 \phi\, d\mathbf{s} = \iint \mathbf{n} \times \operatorname{grad} \phi\, dS,$$

$$\int_0 \mathbf{R} \cdot d\mathbf{s} = \iint \mathbf{n} \cdot \operatorname{curl} \mathbf{R}\, dS,$$

$$\iint \mathbf{n}\phi\, dS = \iiint \operatorname{grad} \phi\, dv,$$

$$\iint \mathbf{n} \cdot \mathbf{R}\, dS = \iiint \operatorname{div} \mathbf{R}\, dv,$$

$$\iint \mathbf{n} \times \mathbf{R}\, dS = \iiint \operatorname{curl} \mathbf{R}\, dv.$$

These forms are very simple, while in comparison their Cartesian equivalents are clumsy.

There is another great gain which would almost certainly follow upon the change under consideration. Looking down the tables of contents in our mathematical journals one is struck with the preponderance of papers in pure over those in applied subjects, and notices how few of the latter are really three-dimensional. We have seen a far greater number of our brilliant students take up the pure in preference to the applied; while many who have a leaning toward the latter choose laboratory work in physics, and in some cases forget the mathematical claims of the subject in their devotion to the experimental. The writer is confident that an early training in vector thought and vector analysis would go far toward inducing a much larger proportion of our students to cultivate the study of physical mathematics, and then to undertake research in this domain. The practice of visualizing inculcated by such a training leads to the same kind of conceptions as those which Faraday formed of the condition of the electric field.*

The place of Vector Analysis in a University course would depend to a certain extent upon the earlier mathematical curriculum. Speaking generally, however, the purely algebraic part of the subject including the different kinds of products of two or more vectors, and perhaps the elementary part of the theory of linear vector functions,† might be given in the first year. A considerable application of this may be made to the theory of elementary mechanics, without any introduction of the infinitesimal calculus. The differential and integral calculus of vectors might be postponed till the second year, where it would still be in plenty of time for use in those applied subjects for which it is so admirably fitted. The teaching of analytical solid geometry, which is often begun at this stage, would be rendered much easier by the vector knowledge thus gained; and many of the proofs in this subject are greatly simplified by vector methods.

In conclusion let it be distinctly stated that *nothing is further from my thoughts than the abolition of Cartesian analysis. Such a thing is impossible.* But this analysis is an incomplete instrument in itself, and that whose adoption I have been pleading goes a long way toward completing it. In some respects the older analysis suggests the reading of a book by spelling every word on each page. In vector analysis we deal with words rather than letters, and in so doing can give more of our attention to the thoughts. But we never forget that in written language the primary elements are the letters; and the vector analyst often finds it instructive to spell his words, resolving his vectors into components and expanding his single equations into Cartesian trios. Thus *the new analysis has not come to destroy the old, but to fulfil it.*

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* Cf. Coffin, *Vector Analysis*, Preface, p. vi. † This is purely algebraic.

NOTE.—Since writing this paper I have read those portions of Heaviside's *Electromagnetic Theory* dealing with vectorial algebra and analysis, and giving incidentally* some idea of the controversy toward the close of the last century for and against the use of quaternions in mathematical physics. I am very glad that before beginning the study of vector analysis I had no knowledge of quaternions beyond the name and a few vague ideas about the controversy referred to. I was thus able to approach the subject quite unbiassed and *from the Cartesian point of view*. This, I believe, makes all the difference to one's reception of the vectorial method; for, as stated at the close of my paper, the Cartesian and vectorial analyses are inseparably connected. Since writing those lines, I was pleased to find the same thought expressed by Heaviside. "The quaternionists," he says, "want to throw away the 'Cartesian trammels,' as they call them. This may do for quaternions, but with vectors would be a grave mistake. My system, so far from being inimical to the Cartesian system of mathematics, is its very essence." †

The unfortunate idea that vector analysis is a sort of modified system of quaternions is perhaps largely responsible for its tardy adoption in Great Britain. The consideration of my own experience may therefore be of some value, chiefly because there were no initial conceptions and associations to bias me. The first book I studied on the subject was the French edition of *Le Calcul Vectoriel*, by Burali-Forti and Marcolongo. I found their system interesting and helpful, though not always natural. While waiting for other books by mail I read Silberstein's *Vectorial Mechanics*, in which Heaviside's notation and analysis are employed. Speaking only of the chapter on vector algebra and analysis, I thought it simple and direct, but found the symbol *V* used to denote a vector product very confusing. I have since, of course, learnt that this is only a survival of the quaternionic notation, and yet adopted, strange to say, by such a pronounced anti-quaternionist as Heaviside. Then I had the pleasure of reading Gibb's *Vector Analysis*, by E. B. Wilson, explaining a system substantially the same as that of Heaviside, from which it differs chiefly in notation. It appeared to me at once most natural in both method and notation, and afforded me the inspiration already referred to. The bracket notation for products of vectors has since come under my notice in many works, but while reading this with equal facility I prefer that of Gibbs. Not only does this leave the brackets free for ordinary algebraic purposes, but the dot and cross are now hardly needed in algebra to indicate a product.

C E. WEATHERBURN.

23rd September, 1916,
Ormond College, Parkville, Melbourne.

THE PRINCIPLES OF PROBABILITY AND APPROXIMATIONS IN ARITHMETIC.

BY W. HOPE-JONES, B.A.

I SUPPOSE that most teachers of Elementary Mathematics, when first they broach the subject of finding the circumference of a circle, write on their blackboards for comparison

$$\pi = 3{\cdot}14159,$$
$$3\tfrac{1}{7} = 3{\cdot}14286,$$

* In a manner often blunt and amusing, but always interesting.

† *Loc. cit.*, vol. i. p. 305.

and draw the conclusion that "as these results agree as far as the third figure, the approximation $\pi = 3\frac{1}{7}$ can reasonably be expected to get the first 3 figures of your answer right." The purpose of this paper is to analyse to some extent a few practical rules of this type.

1. A diameter, chosen at random, and reckoned in any units, is multiplied by $3\frac{1}{7}$, and the result given correct to 3 significant figures. Find the probability that the circumference so obtained is correct to 3 significant figures.

Let the exact circumference, multiplied or divided by some integral power of 10, be c units, c being between 100 and 1000. Then the approximate circumference, similarly multiplied or divided, $=\dfrac{3\frac{1}{7}c}{\pi}$.

The very rare case in which $c<1000$ but $\dfrac{3\frac{1}{7}c}{\pi}>1000$ is left out of account, and terms involving powers of $(3\frac{1}{7}-\pi)$ are omitted. The frequency of c is not uniform, but can be proved inversely proportional to c.

Proof. If x is a circumference "chosen at random," let the probability that x is between X and $X+dX$ be $f(X)dX$.

Then the probability that bx is between bX and $bX+bdX=f(X)dX$, b being any constant.

But this probability is by definition $f(bX)d(bX)$, bX being also "chosen at random."

$$\therefore\ f(X)dX=f(bX)d(bX),\quad \frac{f(bX)}{f(X)}=\frac{1}{b}\quad\text{and}\quad f(X)\propto\frac{1}{X}.$$

That is, the frequency of X is proportional to $\dfrac{1}{X}$, so that the frequency of $c=\dfrac{B}{c}$, B being constant. The probability of a value between X and $X+dX=\dfrac{B\,dX}{X}=B\,d(\log X)$, so that all values of $\log X$ are equally likely. Values of c, in fact, are distributed uniformly along a slide-rule, not along a tape-measure.

In practice it is impossible to find quantities which are chosen perfectly at random: but a rough experimental verification of this result can be found in facts of the following kind. Of the 364 towns and London boroughs mentioned in the Municipal Directory of England and Wales in *Whitaker's Almanack* for 1914, the populations of 108 begin with the figure 1; and $\frac{108}{364}$ $(=0{\cdot}297)$ is a good approximation to $\dfrac{\log 2}{\log 10}$, that is $\dfrac{\int_1^2 \frac{B\,dX}{X}}{\int_1^{10}\frac{B\,dX}{X}}$.

The correctness of the corresponding approximate circumference depends on whether or not an integer$+\frac{1}{2}$ comes between c and $\dfrac{3\frac{1}{7}c}{\pi}$. Hence, for the approximate circumference to be correct, it is necessary that c should not lie within certain "prohibited regions," namely, from $\dfrac{(n+\frac{1}{2})\pi}{3\frac{1}{7}}$ to $(n+\frac{1}{2})$. $(n=100,\ 101, \dots\ 999.)$

The width of the "prohibited region" is proportional to $n+\frac{1}{2}$, and so approximately to c. Strictly speaking, we cannot say that the probability of c falling within a "prohibited region" is proportional to the magnitude of c, because for any given value of c this probability is either exactly 0 or exactly 1: but, taking the *à priori* probability of c falling into a "prohibited region" in any particular neighbourhood as the fraction which "prohibited regions" form of the whole range of c in that neighbourhood, we may say that the probability of an integer$+\frac{1}{2}$ coming between c and $\dfrac{3\frac{1}{7}c}{\pi}$ is $\dfrac{3\frac{1}{7}c}{\pi}-c$, that is, the width of the "prohibited region."

Let the whole probability of the approximate circumference being incorrect to 3 figures be A;

$$\therefore\ A=\int_{100}^{1000}\left(\frac{3\frac{1}{7}c}{\pi}-c\right)(\text{frequency of } c)dc \Big/ \int_{100}^{1000}(\text{frequency of } c)dc$$

$$=\int_{100}^{1000}\left(\frac{3\frac{1}{7}c}{\pi}-c\right)\frac{B\,dc}{c} \Big/ \int_{100}^{1000}\frac{B\,dc}{c}$$

$$=900\left(\frac{3\frac{1}{7}}{\pi}-1\right)\Big/(\log_e 1000-\log_e 100)=900\left(\frac{3\frac{1}{7}}{\pi}-1\right)\log_{10}e=\cdot 15732. \quad \ldots(1)$$

$\therefore$ the common rule that "the approximation $\pi=3\frac{1}{7}$ will generally get the first 3 figures of your answer right" is a sound working rule, because it will succeed in 843 cases out of 1000, chosen at random, supposing the answer to contain π as a factor. The same applies to $\frac{1}{\pi}$; and if the factor is $\pi^{\pm\frac{1}{2}}$, the chance of error will naturally be reduced from $\cdot 157$ to $\cdot 079$.

The result (1) can be generalised as follows:

If, in a quantity containing p as a factor, P be substituted as an approximation for p, $\frac{P\sim p}{p}$ being less than 10^{-n}, the probability that the answer is incorrect to n significant figures is

$$9\,.\,10^{n-1}\left(\frac{P}{p}\sim 1\right)\log_{10}e,\ \textit{i.e.}\ 3\cdot 909\left(\frac{P}{p}\sim 1\right)10^{n-1}. \quad \ldots\ldots\ldots\ldots(2)$$

Another important special case of this is the approximation, 1 cubic foot $=6\frac{1}{4}$ gallons.

Here $P=6\cdot 25$, $p=6\cdot 23211$ (*v.* Chambers's *Mathematical Tables*), and $n=$any integer less than 3. ($n=2$ gives the most useful result.) Then the probability that a quantity, given accurately in cubic feet, will be reduced to gallons, correct to 2 figures, by the $6\frac{1}{4}$ approximation, is

$$1-3\cdot 909\left(\frac{6\cdot 25}{6\cdot 23211}-1\right)\times 10=1-\cdot 112=\cdot 888.$$

By a similar substitution we find that if a number of miles is reduced to Km., by taking 5 miles $=8$ Km., the result will be correct to 2 figures in 772 cases out of 1000.

2. Returning to the approximation $\pi=3\frac{1}{7}$, it is evident that it will sometimes give an answer correct to 4 figures. Let us find how often this will happen.

In the result (1), $\quad A=\int_{100}^{1000}\left(\frac{3\frac{1}{7}c}{\pi}-c\right)B\frac{dc}{c}\Big/\int_{100}^{1000}B\frac{dc}{c}$,

the effect of dealing with the 4th instead of the 3rd figure will be to alter the lower limit in both integrals to 1000, and the upper limit in the denominator to 10,000: but in the numerator, before c reaches 10,000, the width of the "prohibited regions" $\left(\frac{3\frac{1}{7}c}{\pi}-c\right)$ will have increased to 1, so that above that limit c cannot possibly keep out of them, and the approximate answer is bound to be wrong. This begins to occur as soon as $\frac{3\frac{1}{7}c}{\pi}-c$ rises to 1, that is when $c=\frac{\pi}{3\frac{1}{7}-\pi}$.

$\therefore$ the chance that the approximation $\pi = 3\frac{1}{7}$ will give an answer incorrect to 4 figures is

$$\left\{\int_{1000}^{\frac{\pi}{3\frac{1}{7}-\pi}}\left(\frac{3\frac{1}{7}c}{\pi}-c\right)B\frac{dc}{c}+\int_{\frac{\pi}{3\frac{1}{7}-\pi}}^{10000}B\frac{dc}{c}\right\}\Big/\int_{1000}^{10000}B\frac{dc}{c}$$

$$=\left\{\left(\frac{3\frac{1}{7}}{\pi}-1\right)\left(\frac{\pi}{3\frac{1}{7}-\pi}-1000\right)+\log_e 10000-\log_e\left(\frac{\pi}{3\frac{1}{7}-\pi}\right)\right\}\Big/\log_e 10$$

$$=\left\{1-1000\left(\frac{3\frac{1}{7}}{\pi}-1\right)\right\}\log_{10}e+4+\log_{10}\left(\frac{3\frac{1}{7}-\pi}{\pi}\right)=\cdot 864. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

This gives us the complete answer to the question, "Why should I give the answers to circle questions correct to 3 figures and not to 4?" "Because if you try for 3 figures, you will get them right more than 5 times out of 6: but if you try for a 4th figure, it will be wrong more than 6 times out of 7."

The result (3) may be generalised thus:

Defining p and P as before, $\frac{P\sim p}{p}$ being between 10^{-n} and 10^{-n-1}, the probability that the approximate answer is incorrect, if given to $n+1$ figures, is

$$\left\{1-10^n\left(\frac{P}{p}\sim 1\right)\right\}\log_{10}e+(n+1)+\log_{10}\left(\frac{P}{p}\sim 1\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The minimum value of this being $\cdot 3909$, the $(n+1)$st figure is not, in general, trustworthy.

The following table is of value in deciding how many figures we should give up, or teach our pupils to give up, in answers obtained by using certain common approximations:

Approximation.	Probability of obtaining answer correct to 2	3	4 significant figures.
$3\frac{1}{7}$ for π	·984	·843	·136
$\sqrt{3\frac{1}{7}}$ for $\sqrt{\pi}$	·992	·921	·349
$6\frac{1}{4}$ gallons for 1 ft^3.	·888	·232	0
$\frac{5}{8}$ mile for 1 Km.	·772	·053	0
32 for g (London)	·768	·050	0
$\sqrt{32}$ for $\sqrt{g}$ (London)	·884	·223	0

(To be continued.)

MATHEMATICAL NOTES.

500. [X. iv. a.] *Required, a Solution of the following*:

EW is a straight line, Pp is parallel to EW at a distance A; Qq is parallel to Pp at a distance B; Rr is parallel to Qq at a distance C. P, Q, R are any points on Pp, Qq, Rr.

EW represents a road running E and W. The three strips, which we may call A, B, C, are strips of cultivated land, the speeds across which are respectively $1/a$, $1/b$, and $1/c$ of the speed on the road. Speeds equal to that on the road are possible along the three parallel tracks Pp, Qq, Rr.

At a given point on EW an order is received to reach the position P in the shortest time. What is the course? What would be the course were

the order to be given at a given point on Pp to reach Q, and at a given point on Qq to reach R?

(2) What would be the course were the order received at a given point on EW to reach R?

J. D. J. Bishop, Major, Glos. Regt.

501. [V. 1. a. δ.] *"Russian Peasant" Multiplication in Roman Numerals.*

A method of multiplication stated to be in common use among the peasantry of the villages in Russia has been brought into notice in a note recently published in America. The method involves only the operations of doubling, halving and adding, and the object of this note is to show that it can be applied equally well to numbers expressed in Roman numerals.

The method is illustrated by the following example; say 37 × 39.

37	39
18	[78] omit
9	156
4	[312] omit
2	[624] omit
1	1248
Product	1443

The numbers in the first column are got by repeatedly halving one factor, omitting remainders; those in the second column by repeatedly doubling the other factor. The numbers in the second column which stand opposite even numbers in the first are struck out, and the sum of those standing opposite odd numbers gives the required product.

It will be seen that the process of halving the numbers in the first column is exactly the same as would be required to express this factor in the binary scale of notation, only, instead of the remainders being written down, their positions are indicated by the odd numbers in the column. Thus $37 = 100101$ or $2^5 + 2^2 + 1$; and the numbers added in the second column represent 39 multiplied by $1, 2^2, 2^5$ respectively.

When applied to Roman numerals, the process stands thus, for example: Multiply XLVI by LXIII.

XLVI	[LXIII] omit
XXIII	CXXVI
XI	CCLII
V	DIV
II	[MVIII] omit
I	MMXVI
	MMDCCCXCVIII

G. H. Bryan.

502. [C. 1. e. g.] *Note on the Teaching of the Mean Value Theorem and its Extensions.*

Two of the greatest stumbling blocks to the ordinary student of the Differential Calculus are the Mean Value Theorem and Taylor's Theorem. The argument given in Lamb's *Calculus*, § 56, is easy to follow, but there is always difficulty in writing down the function in (2). I suggest that the idea of comparison of $y=f(x)$ with another curve, as in § 66, be used for the Mean Value Theorem. Not only does that course simplify the proof of the Theorem itself, but it forms an easy introduction to the work of §§ 66 and 67, often found difficult. Starting by comparing $y=f(x)$ with $y=Ax+B$, we have the proof of the Mean Value Theorem. Proceeding then to a comparison with $y=Ax^2+Bx+C$, we obtain the results of §§ 66 and 67. The student then finds it fairly easy to extend the method to the proof of Taylor's Theorem as follows:

Writing $$F(x)=f(x)-[A_nx^n+\ldots+A_rx^r+\ldots+A_0],$$
we determine $A_0, A_1, \ldots A_n$ by the $(n+1)$ conditions,
$$F(a)=0,\ F_1(a)=0, \ldots F_{n-1}(a)=0,\ F(a+h)=0.$$
The usual argument gives
$$n!A_n=f_n(a+\theta h).$$
The $(n+1)$ conditions defining $A_0, A_1, \ldots A_n$ give
$$f(a)=A_na^n+\ldots+A_ra^r+\ldots+A_0,$$
$$f_1(a)=nA_na^{n-1}+\ldots+rA_ra^{r-1}+\ldots+A_1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$f_r(a)=\frac{n!}{(n-r)!}A_na^{n-r}+\ldots+r!Ar,$$
$$f_{n-1}(a)=\frac{n!}{1!}A_n(a)+(n-1)!A_{n-1}.$$
Hence $$f(a)+hf'(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f_{n-1}(a)$$
$$=A_0+\ldots+A_r(a+h)^r+\ldots+A_n[(a+h)^n-h^n]$$
$$=f(a+h)-A_nh^n.$$
Substituting the value of A_n, the result follows.

University of Queensland. H. J. Priestley.

503. [V. i. a. δ.] *Arithmetical Accuracy Test.*

Last summer an accuracy test was carried out in sixteen Public Schools of various types.

The papers and a summary of the results are printed below. The time allowed was 40 minutes. Each question was marked right or wrong, and the number who failed to get the right answer was recorded.

The Papers.

A. 1. Add the ten sums of money given below. [£152,328 16s. 9d.

2. Express 200,000 inches as a compound quantity, in mi., yd., ft., in. [3, 275, 1, 8.

3. Find the value of $(23{\cdot}7\times0{\cdot}315)^2$, to three significant figures. [55·7.

4. Express $\dfrac{£2.\ 17s.\ 7d.}{£11.\ 13s.\ 6d.}$ as a decimal, to three significant figures. [0·616.

5. Find, to the nearest sq. ft., the total area of 20 boards, each 8 ft. 6 in. by 11 in. [156 sq. ft.

B. 1. Add the first *nine* sums of money given below. [£146,608. 17s. 1d.

2. Express 200,000 sec. as a compound quantity, in days, hours, etc. [2, 7, 33, 20.

3. Multiply 347 by 0·0365. [12·6655.

4. Divide 2·7 by 514·3, to five decimal places. [0·00525.

5. Multiply $3\frac{1}{5}$ by $2\frac{1}{3}$ and subtract the result from $10\frac{1}{12}$. [$2\frac{37}{60}$.

C. 1. Add the first *six* sums of money given below. [£101,056. 7s. 6d.

2. Reduce 6 t. 12 cwt. to pounds. [14784 lb.

3. Multiply 0·216 by 0·358. [0·077328.

4. Divide 1·63 by 8·54, to three decimal places. [0·191.

5. Multiply $12\frac{3}{7}$ by $5\frac{2}{3}$. [$70\frac{3}{7}$.

D. 1. Add the *last six* sums of money given below. [£98,379. 10s. 8d.

2. Reduce £253. 17s. 3d. to pence. [60927d.

3. Multiply 0·642 by 3·414. [2·191788.

4. Divide 0·815 by 0·673, to two decimal places. [1·21.

5. Divide £1273. 16s. 6d. by 43, to the nearest penny. [£29. 12s. 6d.

(i)	£8643 13 5			(vi)	£40,627 15 6		
(ii)	£27,127 3 10			(vii)	£7119 11 10		
(iii)	£349 18 1			(viii)	£37,826 4 7		
(iv)	£17,828 10 9			(ix)	£606 13 2		
(v)	£6479 5 11			(x)	£5719 19 8		

Summary of Results.

All results shown are the number of mistakes expressed as percentages of the whole number of answers: thus, for School (i) below, the number of mistakes in Qn. 1 (including all the papers, A, B, C, D) was 8 per cent., and of all the answers to the whole of all the papers 10 per cent. were wrong.

School.	Number of answers wrong (as percentages) to QUESTIONS					Mistakes in decimal point (as percentages) in QUESTIONS		Number of mistakes in all questions (as percentages).
	1	2	3	4	5	3	4	
i	8	7	9	12	13	2	3	10
ii	13	18	11	23	20	2	8	17
iii	17	18	18	19	21	4	6	19
iv	22	12	18	23	20	Returns incomplete.		19
v	12	13	21	37	16	3	6	20
vi	20	15	18	25	21	5	6	20
vii	25	16	21	26	31	5	7	24
viii	32	20	18	32	27	7	13	26
ix	23	19	24	36	28	4	11	26
x	18	20	18	45	28	5	16	26
xi	21	13	31	44	27	7	17	27
xii	25	29	19	33	32	7	9	28
xiii	25	26	26	40	34	9	13	30
xiv	33	30	34	40	39	12	11	35
xv	28	26	39	35	49	12	13	35
xvi	29	27	33	53	41	9	20	37

A. W. Siddons.

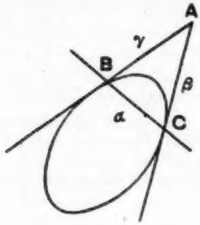
504. [L^1. 6. c.] *On the circles of curvature at B and C of the conic $a^2=\beta\gamma$.*

1. Their equations are respectively

$$b(a^2+\gamma^2+2a\gamma\cos B)=\gamma\,.\,\Sigma a$$

and

$$c(a^2+\beta^2+2a\beta\cos C)=\beta\,.\,\Sigma a.$$



These may be written

$$\Sigma a\beta\gamma=\frac{ac}{b^2}\left(\frac{ba}{a}+\frac{b-c}{c}\beta\right)\Sigma aa$$

and

$$\Sigma a\beta\gamma=\frac{ab}{c^2}\left(\frac{ca}{a}-\frac{b-c}{b}\beta\right)\Sigma aa,$$

so that the radical axis of the circles is $bc(b+c)a=a(b^2\beta+c^2\gamma)$, on reduction.

2. The chords of intersection of the circles with the curve are

$$(a-2b\cos B)a-(b-c)\gamma=0$$

and

$$(a-2c\cos C)a+(b-c)\beta=0, \text{ respectively.}$$

These meet at the point given by

$$\frac{a_0}{b-c}=\frac{\beta_0}{-(a-2c\cos C)}=\frac{\gamma_0}{a-2b\cos B}\equiv\lambda,$$

while

$$a_0^2-\beta_0\gamma_0\equiv 2\lambda^2(a^2-bc)(1-\cos A).$$

3. *If* $a^2=bc$, the circles will intersect on the curve itself; and the radical axis will become $a(b+c)a=b^2\beta+c^2\gamma$, thus passing through the point $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ or the C.G. of the triangle of reference.

Moreover, the radical axis cuts the curve where $a^2(b+c)^2\beta\gamma=(b^2\beta+c^2\gamma)^2$ or $(b^3\beta-c^3\gamma)(b\beta-c\gamma)=0$.

Thus the common point of the circles and the curve is $\left(\frac{1}{a^3}, \frac{1}{b^3}, \frac{1}{c^3}\right)$.

The other point of intersection of the circles is $(2a\cos A,\ b,\ c)$, which is collinear with the vertex A and the symmedian point (a, b, c).

4. Note on the point $\left(\frac{1}{a^3}, \frac{1}{b^3}, \frac{1}{c^3}\right)$ in any triangle.

Let the above point be called P, and the Brocard points Ω and Ω'.

Also let $\lambda\equiv 2\Delta/\Sigma b^2c^2$.

Then the actual perpendiculars on the sides of the triangle of reference are:

for Ω,	$c^2a\lambda$,	$a^2b\lambda$,	$b^2c\lambda$,
for Ω',	$ab^2\lambda$,	$bc^2\lambda$,	$ca^2\lambda$,
for P,	$\frac{b^2c^2}{a}\lambda$,	$\frac{c^2a^2}{b}\lambda$,	$\frac{a^2b^2}{c}\lambda$.

By addition of the columns it is evident that the C.G. of the triangle of reference is also the C.G. of the triangle $P\Omega\Omega'$. Hence the point P is easily constructed.

N. M. GIBBINS.

505. [L¹. 2 a, b, c.] *A Case of Three Rotating Lines and the Point "O."*

Perhaps you might allow me to add the following to the interesting paper in connection with the circle on GH the locus of "O" (vol. vi. page 392).

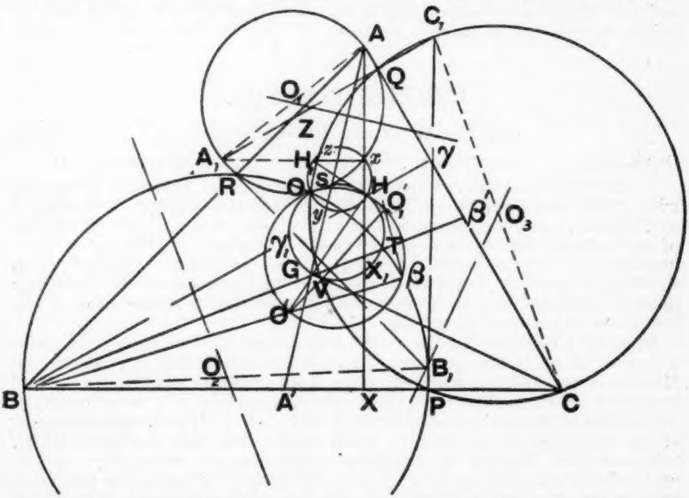
I. (1) Let O_1, O_2, O_3 be the three circles of the three coaxial systems used to determine P, Q and R.

On O_1 take any point A_1. Join A_1R, A_1Q cutting O_2O_3 in B_1 and C_1; then $A_1B_1C_1$ is a triangle through P, Q and R similar to ABC; $\angle A_1G_1B_1 = \angle AGB$, and so on. And if AG cut O_1 in S, BG cut O_2 in T, and CG cut O_3 in V, the angles SA_1R and SAR are equal, and so for BT and B_1T; therefore A_1S meets B_1T at a constant angle equal SGT at G_1, *i.e.* $\angle SG_1T = \angle SGT$; therefore the circle SGT *is the locus of the centroid* of all the series of triangles $A_1B_1C_1$.

(2) When A_1 is at O the triangle vanishes.

When A_1 is at opposite end of diameter of O_1 through O, $A_1B_1C_1$ is a maximum, for ORA_1, etc., are right angles.

(3) When A_1 is at end of diameter of O_1 through A, every line in the triangle $A_1B_1C_1$ is at right angles to its corresponding line in ABC, for A_1RA is a right angle, and so for B_1 and C_1; any two points, say A_2, A_3, at opposite ends of any diameter of O_1 will give triangles at right angles in this respect.



(4) The locus of H the orthocentre is an orthogonal circle to GOH through the points x, y, z, where the perpendiculars to AB, etc., cut O_1,O_2,O_3; for HOH_1 is a right angle, and HOG is a right angle; hence GOH_1 is a straight line and GHH_1 is a right angle.

(5) The locus of O' the circumcentre is a circle coaxial with the G and H circles, for it must pass through O the minimum centre, and it passes through H, for $O'HO_1'$ is a right angle. To find O_1' take β on O_2 where cut by BO', and then $B_1\beta$ at right angles to $B\beta$ cuts H_1H at O_1'.

(6) The locus of the centre of the nine-point circle is also easily shown to be another coaxial of the group.

(7) The loci of the inscribed and escribed circle centres I, I_1, I_2, I_3 are also easily seen to be circles passing through O and forming six coaxial

triads, for if the bisectors of A, B, and C cut O_1, O_2, O_3, in l, m, n, and the external bisectors in k_1, k_2, k_3, then the locus of I passes through l, m, and n; the locus of I_1 passes through k_2 and k_3, and we have loci of I, I_2, and O_1 passing through O and l, and so on for first triad; in second triad I_1, I_2, and O_2 pass through O and k_2, and so on.

II. (1) Instead of forming our triad of coaxials by bisecting AH, etc., and taking line of centres at right angles to AG, etc., form a triad of systems by bisecting AG, etc., and take the lines of centres at right angles to AH, etc., through the point of bisection, and we get as the locus of O the same circle on GH as diameter, but it is *now the locus of the orthocentre* of a series of triangles taken as before, a vertex on each circle, and sides through P_1, Q_1, and R_1. This may be seen from the fact that the common points of the coaxials will be A and X_1, B and Y_1, C and Z_1.

(2) This is one of a series of propositions regarding fixed linear points connected with the triangle; for instance, if O' is the circumcentre, bisecting AH as before and taking a line of centres at right angles to AO', through the point of bisection, we get as the locus of O a fixed circle on $O'H$, which is the locus of the *circumcentres* of a series of triangles formed as above; or bisecting AO', and taking a line of centres at right angles to AH, we get the same circle as locus of O, but it is now the locus of the *orthocentres* of the series of triangles connected with it as above; and so on for other pairs of points, each having its coaxials attached. WILLIAM FINLAYSON.

CORRESPONDENCE.

TO THE EDITOR OF THE *Mathematical Gazette*.

DEAR SIR, *November*, 1916.

The Teaching Committee of the Mathematical Association concurs with the Councils of the Classical, English, Geographical, Historical and Modern Language Associations in the view that any reorganisation of our educational system should make adequate provision for both humanistic and scientific studies; that premature specialisation should be avoided; and that technical preparation for a particular profession should be conceived in such a spirit that it misses none of the essentials of a liberal education.

In reply to the invitation of the representative Conference to make a statement as to the position of mathematical studies in schools, the Mathematical Association Committee would submit that from a school course of mathematics the pupil should acquire: (1) an elementary knowledge of the properties of number and space; (2) a certain command of the methods by which such knowledge is reached and established, together with facility in applying mathematical knowledge to the problems of the laboratory and the workshop; (3) valuable habits of precise thought and expression; (4) some understanding of the part played by mathematics in industry and the practical arts, as an instrument of discovery in the sciences and as a means of social organisation and progress; (5) some appreciation of organised abstract thought as one of the highest and most fruitful forms of intellectual activity.

(Signed) A. N. WHITEHEAD (*President*),
A. W. SIDDONS (*Chairman of the Teaching Committee*),
On behalf of the Mathematical Association.

(Any communication with regard to the above may be addressed to A. W. SIDDONS, Harrow School.)

UNIVERSITY COLLEGE, GOWER STREET, LONDON, W.C.,
11th December, 1916.

DEAR SIR,

The letter from Prof. Lodge and the late Mr. Jackson, p. 311 of the *Gazette*, raises a number of points.

With regard to the first and second paragraphs of the letter, the difference between the writers and myself is that they in effect deny that the meaning of the expression $a \times b$ can be affected by what precedes it. I am unable to agree with them, and there is no more to be said. My view of the matter is stated on p. 284.

In the third paragraph the writers say that "the whole question before us is whether $a+b\times c$ is or is not ambiguous under present conventions." It appears to me that a great deal more is involved, as I have endeavoured to show on pp. 282-283.

In particular it is absolutely necessary, if it is permissible to omit brackets, to explain in what manner the sign of division is to be dealt with, and I do not therefore understand why the writers in the sixth paragraph charge me with "dragging a red herring across the trail by discussing the evaluation of a *term* in which division as well as multiplication occurs."

And after all, what are the present conventions? The matter is mentioned in the fourth paragraph, but is dealt with more at length in Mr. Jackson's letter on p. 246 (endorsed by Prof. Lodge on p. 247). Mr. Jackson says that he has looked "somewhat carefully into the conventions as to the sequence in which arithmetical operations are to be performed. Many arithmetical books, in their chapter on Fractions, lay down three rules of interpretation. Stated in their baldest form these rules are:

"1. Multiplications and divisions must be performed before additions and subtractions.

"2. Multiplications and divisions must be performed in order (from left to right).

"3. The word 'of' is, however, equivalent to a bracket.

"It will be convenient to state at once the conclusions I have reached, before entering into the arguments on the subject. These are, that the Rule 1, though not always happily expressed, is a rule of fundamental importance, and is essential to the harmony of arithmetic and algebra; but that Rules 2 and 3 are of an artificial character, that they are not necessary and cannot be defended."

On p. 281 I showed that Rule 1 cannot stand if Rule 2 is abandoned, a fact which Prof. Lodge and Mr. Jackson have not yet disproved. In view of what they now say in their sixth paragraph I ask Prof. Lodge to explain to the readers of the *Gazette* how Rule 1 can stand if Rule 2 be abandoned. If there is any flaw in my reasoning on p. 281, will Prof. Lodge kindly explain it? This is a very simple and unambiguous question, and I submit that I am entitled to an answer. If he cannot show that I am wrong, the claim made by Mr. Jackson and himself to clearness in treating this subject cannot be upheld.

The fifth and seventh paragraphs assert the existence of a convention alleged to be firmly established for more than a century. Whether that is the case or not I do not know, and the point is immaterial, for an appeal to tradition reads very strangely in a mathematical journal. The whole of the argument in these paragraphs falls to the ground unless Prof. Lodge is able to answer the questions I have put to him.

The whole matter is one of great gravity in the teaching of Arithmetic. The teacher who uses unnecessary rules develops in the minds of his pupils a spirit which is antagonistic to progress. During upwards of

thirty years of experience in teaching Mathematics I have never used the convention or conventions alleged to exist. I regard them as absolutely useless, and under no circumstances would I inflict them on my pupils. M. J. M. Hill.

THE LIBRARY.

Change of Address.'

The Library is now at 9 Brunswick Square, W.C., the new premises of the Teachers' Guild.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Scarce Back Numbers.

Reserves are kept of A.I.G.T. Reports and Gazettes, and, from time to time, orders come for sets of these. We are now unable to fulfil such orders for want of certain back numbers, which the Librarian will be glad to buy from any member who can spare them, or to exchange other back numbers for them:

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A.I.G.T. Report No. 11 (very important).
A.I.G.T. Reports, Nos. 10, 12.

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